Examples

① Consider $C(\mathbb{R}) = \{\text{continuous fcns } f:\mathbb{R}\to\mathbb{R}\}$.

Let $I_n = \{f:\mathbb{R}\to\mathbb{R} : f(x)=0 \text{ if } x\in[-\frac{1}{n},\frac{1}{n}]\}$ for all $n\in\mathbb{N}$.

$I_n \subseteq I_{n+1}\ \forall n$

It is never the case that $I_n = I_{n+1}$,

So $C(\mathbb{R})$ is not Noetherian, because it does not satisfy the ascending chain condition (ACC)

② Observe that $\mathbb{Z}$ satisfies the ACC.

For any ideal in $\mathbb{Z}$, since it's a principal ideal domain, every ideal I satisfies $I = \langle n\rangle$, for some $n\in\mathbb{Z}$.

If J is an ideal that contains I, then $J = \langle k\rangle$, where $k \mid n$.

Since there are only a finite # of divisors of each n, there cannot be an ideal sequence

$I_1 \subsetneq I_2 \subsetneq \dots$ — i.e. eventually the sequence must stabilize, so that $\mathbb{Z}$ satisfies ACC!

Facts: • If R is a PID, then automatically it satisfies ACC.

Sketch of proof: Suppose we have a sequence of ideals $I_1 \subseteq I_2 \subseteq I_3 \subseteq \cdots$,

Let $I_\infty := \bigcup_{n \in \mathbb{N}} I_n$.

- Can prove I_∞ is an ideal, so

$$I_\infty = \langle x \rangle \text{ for some } x.$$

$\rightsquigarrow$ Show $x \in I_n$ for some n.

$\Rightarrow$ the chain stabilizes .... ✓.

Consequence: Sp. R is a PID, $a \in R$ is a nonunit that is nonzero. Then

$$a = \text{product of irreducibles}.$$

Sketch of proof.

- Show a has an irreducible factor - use ACC.

If $a = a_1 b_1 \Rightarrow a \in \langle a_1 \rangle \cap \langle b_1 \rangle = \langle c_1 \rangle$

nonunits

Since b_1 is not a unit $\Rightarrow \langle a \rangle \subsetneq \langle a_1 \rangle$.

Keep going - show $a = a_k \cdot B$, where a_k is irreducible.

- $a = p_1 b_1 = p_1 p_2 b_2 \ldots$

irred irred use ACC. ✓

Consequence: If R is a PID, then R is a UFD.

Sketch of proof: If $a \in R$,

Sp. $a = p_1 \cdots p_r = q_1 \cdots q_s$

$s > r$.

use properties of primes to show

$p_1 \mid q_1 \cdots q_s \mid$

--- keep going .....

<u>Fact</u>: Every Euclidean domain (ED) is a PID.

<u>Sketch of Proof</u>: Sp. R is an ED. Let I be an ideal in R. Choose $u \in I \setminus \{0\}$ s.t. $d(u)$ is minimal, where $d: R \to \mathbb{N} \cup \{0\}$ is the Euclidean function.

$\forall\, a \in I$, use the division algorithm to show that $a \in \langle u \rangle$.

ID $\supsetneq$ UFD $\supsetneq$ PID $\supsetneq$ ED $\supsetneq$ Field $\supsetneq$ Algebraically closed fields.

$d: R \to \mathbb{N} \cup \{0\}$

$\mathbb{Z}[\sqrt{-3}]$ is an ID, not a UFD.

$\mathbb{Q}[\sqrt{5}]$ a UFD, not a PID.

$\mathbb{Z}[x]$ a UFD, not a PID.

$\mathbb{Q}[\sqrt{-19}]$ a PID, not an ED

If $\mathbb{F}$ is a field, $\mathbb{F}[x]$ is a E.D, but not a field.

$\mathbb{Z}$ is a E.D., not a field.

$\mathbb{R}$ is a field, but is not algebraically closed.

$\mathbb{C}$ is an algebraically closed field.